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Common Cause Completability of Classical and Quantum Probability Spaces†

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It is shown that for a given set of correlations either in a classical or in a quantum probability space both the classical and the quantum probability spaces are extendable in such a way that the extension contains common causes of the given correlations, where common cause is taken in the sense of Reichenbach's definition. These results strongly restrict the possible ways of disproving Reichenbach's common cause principle and indicate that EPR-type quantum correlations might very well have a common cause explanation.

1. THE PROBLEM

The aim of this paper is to present two results on the following problem, raised first within the framework of classical, Kolmogorovian probability theory in ref. 4, Chapter 14: Let $(\mathcal{L}, p)$ be a generalized probability space with the orthomodular lattice $\mathcal{L}$ and additive, normalized measure p on $\mathcal{L}$ and let $\{(A_i, B_i) | i \in I\}$ be a set of events in $\mathcal{L}$ that are (positively) correlated with respect to p, i.e., $p(A_i \wedge B_i) > p(A_i)p(B_i)$, with A_i and B_i being compatible for every i. Assume, furthermore, that there exists no element C_i in $\mathcal{L}$ that can be considered the common cause of the correlation between A_i and B_i in the sense of Reichenbach's definition of common cause (see Definition 1 below). The problem is whether $(\mathcal{L}, p)$ can be extended to a probability space $(\mathcal{L}', p')$ in such a way that for every i the extension $\mathcal{L}'$ already contains a

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common cause C_i of the correlation $p(A_i \wedge B_i) > p(A_i)p(B_i)$. If, for a given set of correlations, there exists an extension with the said property, then we call $(\mathcal{L}, p)$ *common cause completable with respect to the set* $\{(A_i, B_i)|i \in I\}$. We have the following result: $(\mathcal{L}, p)$ is common cause completable with respect to the set $\{(A_i, B_i)|i \in I\}$ in the following two cases: (1) $\mathcal{L}$ is a Boolean algebra, p is a classical probability measure on $\mathcal{L}$, and I is finite; (2) $\mathcal{L}$ is a von Neumann lattice, p is a normal state on $\mathcal{L}$, and $\{(A_i, B_i)|i \in I\}$ is the set of all pairs of events that are correlated in p. In fact, we prove more: we show that even if one requires the common cause to satisfy additional constraints formulated in terms of the probabilities of the events involved, if these additional probabilistic constraints are compatible with the Reichenbach conditions, then there exist extensions containing common causes satisfying the further constrains (see Definition 5 for a precise definition of common cause completability and Proposition 2 and 3 for the results.)

In Section 3 we interpret these two propositions from the point of view of the alleged violation of Reichenbach's common cause principle by quantum mechanics. Our conclusion will be that the standard proofs of violation of the common cause principle by quantum theory contain extra assumptions that are not part of the common cause principle, and that the common cause principle might very well be compatible with the existence of certain observed quantum correlations between spacelike-separated quantum events.

2. REICHENBACH'S NOTION OF COMMON CAUSE

Let $\mathcal{L}$ be an orthomodular (σ-) lattice (σ-lattice) and p be an additive (σ-additive if $\mathcal{L}$ is a σ-lattice) state on $\mathcal{L}$. Two elements $A, B \in \mathcal{L}$ are called *compatible*, $c(A, B)$ in notation, if $A = (A \wedge B) \vee (A \wedge B^{\perp})$. If A, B are compatible and

$$p(A \wedge B) > p(A)p(B) \tag{1}$$

then A and B are called (positively) correlated with respect to the state p.

Definition 1. If A and B are positively correlated, then $C \in \mathcal{L}$ is called a *common cause* of the correlation (1) if C is compatible with both A and B and the following conditions hold:

$$p(A \wedge B|C) = p(A|C)p(B|C) \tag{2}$$

$$p(A \wedge B|C^{\perp}) = p(A|C^{\perp})p(B|C^{\perp}) \tag{3}$$

$$p(A|C) > p(A|C^{\perp}) \tag{4}$$

$$p(B|C) > p(B|C^{\perp}) \tag{5}$$

where $p(X|Y) = p(X \wedge Y) / p(Y)$ denotes the conditional probability of X on condition Y and it is assumed that none of the probabilities $p(X)$ ($X = A, B, C, C^{\perp}$) is equal to zero. The common cause C is called *proper* if it differs from both A and B by more than a p-probability zero event.

The above definition of common cause reduces to that of Reichenbach [6] in the case when $\mathscr{L}$ is a Boolean algebra and p is a classical probability measure on $\mathscr{L}$.

Given a statistically correlated pair of events A, B in a probability space $(\mathscr{L}, p)$, a proper common cause C in the sense of Reichenbach's definition does not necessarily exist in $\mathscr{L}$. If this is the case, then we call $(\mathscr{L}, p)$ *common cause incomplete*. The existence of common cause incomplete probability spaces leads to the question of whether such probability spaces can be enlarged so that the larger probability space contains a proper common cause of the given correlation. What is meant by "enlargement" here is contained in the following definition.

Definition 2. The probability space $(\mathscr{L}', p')$ is called an *extension* of $(\mathscr{L}, p)$ if there exists an embedding $h: \mathscr{L} \to \mathscr{L}'$ such that

$$p(X) = p'(h(X)) \qquad \text{for all} \quad X \in \mathscr{L} \tag{6}$$

Recall that $h: \mathscr{L} \to \mathscr{L}'$ is an embedding if h preserves all lattice operations and $X \neq Y$ implies $h(X) \neq h(Y)$.

This definition of enlargement, and in particular the condition (6), implies that if $(\mathscr{L}'p')$ is an extension of $(\mathscr{L}, p)$ (with respect to the embedding h), then every single correlation $p(A \wedge B) > p(A)p(B)$ in $(\mathscr{L}, p)$ is carried over intact by h into the correlation

$$\begin{aligned} p'(h(A) \wedge h(B)) &= p'(h(A \wedge B)) \\ &= p(A \wedge B) > p(A)p(B) = p'(h(A))p'(h(B)) \end{aligned}$$

Hence, it makes sense to ask whether a correlation in $(\mathscr{L}, p)$ has a Reichenbachian common cause in the extension $(\mathscr{L}', p')$.

Given a correlation $p(A \wedge B) > p(A)p(B)$, we call a set of five real numbers r_C, $r_{A|C}$, $r_{B|C}$, $r_{A|C^{\perp}}$, $r_{B|C^{\perp}}$ *admissible* if they satisfy the conditions

$$0 \leq r_{A|C}, r_{B|C}, r_{A|C^{\perp}}, r_{B|C^{\perp}} \leq 1 \tag{7}$$

$$p(A) = r_{A|C}\, r_C + r_{A|C^{\perp}}\, (1 - r_C) \tag{8}$$

$$p(B) = r_{B|C}\, r_C + r_{B|C^{\perp}}\, (1 - r_C) \tag{9}$$

$$p(A \wedge B) = r_{A|C}\, r_{B|C} r_C + r_{A|C^{\perp}}\, r_{B|C^{\perp}}\, (1 - r_C) \tag{10}$$

$$0 < r_C < 1 \tag{11}$$

$$r_{A|C} > r_{A|C^{\perp}} \tag{12}$$

$$r_{B|C} > r_{B|C^{\perp}} \tag{13}$$

It is easy to see that the above conditions are equivalent with the Reichenbach conditions in the sense that given a correlation $p(A \wedge B) > p(A)p(B)$, the admissible numbers r_C, $r_{A|C}$, $r_{B|C}$, $r_{A|C^{\perp}}$, $r_{B|C^{\perp}}$ are numbers that can be equal with the probabilities that are indicated by their subscripts—provided there exists a common cause C of the correlation; conversely, given a correlation $p(A \wedge B) > p(A)p(B)$, if there exist a C in the set of events such that the numbers $r_C = p(C)$, $r_{A|C} = p(A|C)$, $r_{B|C} = p(B|C)$, $r_{A|C^{\perp}} = p(A|C^{\perp})$, $r_{B|C^{\perp}} = p(B|C^{\perp})$ satisfy (7)–(13), then C is a common cause of the correlation in the sense of Reichenbach.

Definition 3. A common cause C of a correlation $p(A \wedge B) > p(A)p(B)$ is said to have (be of) the *type* $(r_C, r_{A|C}, r_{B|C}, r_{A|C^{\perp}}, r_{B|C^{\perp}})$ if these numbers are equal to the probabilities indicated by the indices, i.e., if the following equations hold:

$$p(C) = r_C \tag{14}$$

$$p(A|C) = r_{A|C} \tag{15}$$

$$p(A|C^{\perp}) = r_{A|C^{\perp}} \tag{16}$$

$$p(B|C) = r_{B|C} \tag{17}$$

$$p(B|C^{\perp}) = r_{B|C^{\perp}} \tag{18}$$

Elementary algebraic calculation shows that the following proposition is true.

Proposition 1. Given any correlation $p(A \wedge B) > p(A)p(B)$ in $(\mathcal{L}, p)$, there exists a nonempty two-parameter family of numbers

$$r_C(t, s),\ r_{A|C}(t, s),\ r_{B|C}(t, s),\ r_{A|C^{\perp}}(t, s),\ r_{B|C^{\perp}}(t, s)$$

that satisfy the relations (7)–(13).

Definition 4. We say that $(\mathcal{L}', p')$ is a type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^{\perp}}, r_{B|C^{\perp}})$ *common cause completion of* $(\mathcal{L}, p)$ *with respect to the correlated events* A, B if $(\mathcal{L}', p')$ is an extension of $(\mathcal{L}, p)$, and there exists a Reichenbachian common cause $C \in \mathcal{L}'$ of type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^{\perp}}, r_{B|C^{\perp}})$ of the correlation $p'(h(A) \wedge h(B)) > p'(h(A))p'(h(B))$.

We can now give the basic definition of the paper:

Definition 5. Let $(\mathcal{L}, p)$ be a probability space and $\{(A_i, B_i) | i \in I\}$ be a set of pairs of correlated events in $\mathcal{L}$. We say that $(\mathcal{L}, p)$ is *common cause completable with respect to the set* $\{(A_i, B_i) | i \in I\}$ *of correlated events* if, given any set of admissible numbers $(r^i_C, r^i_{A|C}, r^i_{B|C}, r^i_{A|C^\perp}, r^i_{B|C^\perp})$ for every $i \in I$, there exists a probability space $(\mathcal{L}', p')$ such that for every $i \in I$ the space $(\mathcal{L}', p')$ is a type $(r^i_C, r_{A|C}, r^i_{B|C}, r^i_{A|C^\perp}, r^i_{B|C^\perp})$ common cause extension of $(\mathcal{L}, p)$ with respect to the correlated events A_i, B_i.

We are in the position to formulate the following problem:

Problem. Is every probability space $(\mathcal{L}, p)$ common cause completable with respect to any set of events that are correlated in p?

The general solution of this problem is not known; however, we have results in two typical cases. These results are formulated in the next two propositions.

Proposition 2. Every classical probability space $(\mathcal{S}, \mu)$ with the Boolean algebra $\mathcal{S}$ and classical probability measure μ is common cause completable with respect to any *finite* set of correlated events.

Proposition 3. Every quantum probability space $(\mathcal{P}(\mathcal{M}), \phi)$ with the von Neumann lattice of projections $\mathcal{P}(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ and a normal state ϕ is common cause completable with respect to the set of pairs of events that are correlated in the state ϕ.

We omit the lengthy and tedious proofs of these two propositions (for a detailed proof see ref. 3).

3. COMMENTS ON THE SIGNIFICANCE OF COMMON CAUSE COMPLETABILITY

Reichenbach's common cause principle is a nontrivial metaphysical claim about the causal structure of the physical world: if a direct causal influence between the probabilistically correlated events A and B does not exist, then there exists a common cause of the correlation (in Reichenbach's sense). One of the difficulties in interpreting quantum mechanics is the alleged impossibility of a common cause explanation of certain (EPR) correlations between spacelike-separated quantum events. If a common cause means exactly the Reichenbachian common cause as specified in Definition 1 and an explanation of the quantum correlations in question is indeed provably impossible in terms of such a common cause, this would indeed falsify Reichenbach's common cause principle. We interpret Propositions 2 and 3 as strong restrictions on the possible proofs aiming to show that common causes of correlations do not exist: any such proof must require of the common cause to satisfy some supplementary conditions beyond and above the Reichenbachian ones (2)–(5); furthermore, those additional conditions clearly cannot be formulated purely in terms of the probabilities $p(C)$, $p(A|C)$,

$p(B|C)$, $p(A|C^{\perp})$, and $p(B|C^{\perp})$. This is because the assumptions in Propositions 2 and 3 contain no restrictions whatsoever on these probabilities—beyond the Reichenbach conditions.

One possible supplementary condition could in principle be to assume that different correlations have the *same* common cause. Note that neither Proposition 2 nor Proposition 3 claims that there exist extensions containing *common* common causes, i.e., common causes shared by two or more members of the given set of correlations. In fact, it is not difficult to show that there exist *classical* probability spaces containing two distinct correlations that *cannot* have a common common cause. It is not surprising, then, that the same holds in the case of quantum correlations, and it is this fact that the standard proofs of impossibility of common causes of EPR correlations prove [e.g., 9, 2, 7]. But there does not seem to be any obvious reason why common causes should also be common common causes, whether of quantum or of any other sort of correlations. In our interpretation of Reichenbach's notion of common cause there is nothing that would justify such an assumption.

One way of going beyond the Reichenbach conditions in the EPR situation is to express "no conspiracy" in terms of (conditional) probabilities involving also events such as the events of choosing the measurements in the two wings of the experimental setup. A detailed investigation in this direction is carried out in ref. 8. The (numerical) results obtained so far are in line with the conclusion of the present paper: a (hidden) common cause explanation of the EPR correlations seems possible.

Yet another way to amend the Reichenbach conditions is to link the problem of common cause explanation of correlations to an underlying nonprobabilistic spacetime causal structure. This is done in ref. 5 in the framework of quantum field theory, where the correlated events belong to well-defined spacetime regions by their construction, hence the common cause can be required to belong to the common causal past of the correlated events. Under this specification it is not even known whether the probability space $(\mathcal{L}, p)$ defined by quantum field theory is common cause incomplete.

It should be mentioned that while the impossibility of (nonprobabilistic) *common* common causes of the (nonprobabilistic) GHZ correlations has been proved in ref. 1, it remains open in that paper whether noncommon common causes of the GHZ correlations exist. It might very well be that noncommon common causes of quantum correlations do indeed exist.

It would be interesting to know if Proposition 2 is true also in the case of an infinite set of correlations. Another open question is whether common cause closed probability spaces exist, where $(\mathcal{L}, p)$ is said to be common cause closed if for any correlation $p(A \wedge B) > p(A)p(B)$ with $A, B \in \mathcal{L}$ there exists a common cause $C \in \mathcal{L}$.

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